The Rogers-Ramanujan Identities

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1 Introduction

A *partition* of a nonnegative integer n is defined to be a non-decreasing sequence of integers that sum to n. We define the *partition function* $p(n)$ to be the number of partitions of the integer n, and by convention we let $p(0) = 1$.

Example 1 (p(4) = 5)**.** The five partitions of 4 are:

$$\begin{aligned} 1+1+1+1 &= 4 \\ 1+1+2 &= 4 \\ 1+3 &= 4 \\ 2+2 &= 4 \\ 4 &= 4 \end{aligned}$$

We call the integers that sum to n the *parts* of the partition. Throughout this paper, we are often interested in the number of partitions that satisfy some condition, for example partitions with odd parts. For this, we use the notation $p(n \mid \text{condition})$. A large number of surprising identities arise among the number of partitions subject to certain constraints, two of which are the Rogers-Ramanujan identities. These can be stated

$$(n \mid \text{parts } \equiv \pm 1 \pmod 5) = p(n \mid \text{2-distinct parts}), \tag{1.1}$$

and

$$(n \mid \text{parts } \equiv \pm 2 \pmod 5) = p(n \mid \text{2-distinct parts, parts} \geq 2). \tag{1.2}$$

Here, 2-distinct parts means that every part differs by at least 2. For example, the only partitions of 4 with 2-distinct parts are the partitions $1 + 3$ and 4. It is easy to verify that these results hold for the case $n = 4$, but in order to prove partition identities we need our results to hold for all values of n.

There are multiple techniques that can be used to extend results such as this one to all values of n. In this paper, we will present two of these techniques, along with some concepts important to integer partitions. We will then provide a proof overview of the Rogers-Ramanujan identities.

2 Bijective Proofs

To prove partition identities, we could attempt to verify the number of partitions subject to our constraints are equal for every value of n, as we did when $n = 4$. However, since n ranges to infinity, this is not possible. Naturally, we could try to find an explicit expression for both sides. But without any simple function such as a polynomial in n for the number of partitions, this approach fails.

Instead, to show that the number of partitions on the left-hand side is equivalent to the number of partitions on the right-hand side for all n, we can show that every partition on the left side of the identity is paired with a unique partition from the right and vice versa. Such a one-to-one pairing between two sets of partitions is a bijection. Hence, to prove a partition identity, we just need to find a bijection between partitions.

Euler was the first known mathematician to discover and prove a partition identity, which is stated as follows:

$$p(n \mid \text{odd parts}) = p(n \mid \text{distinct parts}) \text{ for } n \geq 1. \tag{2.1}$$

A bijection for Euler's identity must transform a partition of odd parts into a partition of distinct parts. The inverse must do the converse.

From odd to distinct parts: Distinct means that there can be at most one of each part. Thus, whenever there are two of any part, we can merge them into one of double size. We repeat this procedure until all parts are distinct.

$$\begin{aligned} 1+1+1+1+1+3+3 &\mapsto 1+(1+1)+(1+1)+(3+3) \\ &\mapsto 1+2+2+6 \\ &\mapsto 1+(2+2)+6 \\ &\mapsto 1+4+6 \end{aligned}$$

From distinct to odd parts: The inverse of merging two equal parts is splitting an even part into two halves. We repeatedly split even parts until only odd parts remain.

$$\begin{aligned} 1+4+6 &\mapsto 1+(2+2)+(3+3) \\ &\mapsto 1+2+2+3+3 \\ &\mapsto 1+(1+1)+(1+1)+3+3 \\ &\mapsto 1+1+1+1+1+3+3 \end{aligned}$$

Example 2. For $n = 8$, the bijective correspondence for Euler's identity is as follows:

$$\begin{aligned} 1+7 &\leftrightarrow 1+7 \\ 3+5 &\leftrightarrow 3+5 \\ 1+1+1+5 &\leftrightarrow 1+2+5 \\ 1+1+3+3 &\leftrightarrow 2+6 \\ 1+1+1+1+1+3 &\leftrightarrow 1+3+4 \\ 1+1+1+1+1+1+1+1 &\leftrightarrow 8 \end{aligned}$$

Unfortunately, attempts to find a simple bijective correspondence for the Rogers-Ramanujan identities fails. While combinatorial proofs of this identity exist, they require much more complicated methods. In order to prove this result we need to introduce another powerful tool used to study partitions.

3 Generating Functions

Although we provided a bijective proof for Euler's identity, Euler primarily represented integer partitions using generating functions. The idea behind generating function uses the following fundamental principle of algebra:

$$q^r \cdot q^s = q^{r+s}.$$

This idea can be used in integer partition as follows. Suppose $S = \{n_1, n_2, n_3\}$ is a set of positive integers. Then

$$(1+q^{n_1})(1+q^{n_2})(1+q^{n_3}) = 1 + q^{n_1} + q^{n_2} + q^{n_3} + q^{n_1+n_2} + q^{n_1+n_3} + q^{n_2+n_3} + q^{n_1+n_2+n_3} \quad (3.1)$$

exhibits in the exponents all the possible partitions using distinct elements of S. The coefficient of q^n is the number of such partitions of n.

Example 3. Let $S = \{2, 4, 6\}$. Then the polynomial from eq. (3.1) is

$$1 + q^2 + q^4 + 2q^6 + q^8 + q^{10} + q^{12}$$

This function is the generating function for partitions in distinct elements of $\{2, 4, 6\}$ and the coefficient of q^6, 2, represents the number of such partitions of 6 ($2 + 4$ and 6).

If we extend S to be the set of all positive integers, then we can conclude that the generating function for $p(n \mid \text{distinct parts})$ is

$$\sum_{n=0}^{\infty} p(n \mid \text{distinct parts}) q^n = \prod_{n=1}^{\infty} (1 + q^n). \quad (3.2)$$

If we instead wish to create a generating function that allows for us to include multiple copies of the element n, we would have the generating function $1 + q + q^n + q^{2n} + q^{3n} + \ldots$

Example 4. The generating function for the number of ways of changing n cents into pennies, nickels, and dimes is

$$(1 + q + q^2 + q^3 + \ldots)(1 + q^5 + q^{10} + q^{15} + \ldots)(1 + q^{10} + q^{20} + q^{30} + \ldots).$$

Using the following property of the sum of an infinite geometric series,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| \leq 1,$$

The generating function can be rewritten as

$$\frac{1}{(1-q)(1-q^5)(1-q^{10})},$$

where $|q| < 1$.

We can again extend this result to all integers to get the generating function for $p(n)$. We have

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \quad (3.3)$$

where $|q| < 1$. Throughout the remainder of this paper we will assume $|q| < 1$.

3.1 Euler's Identity Revisited

Recall from eq. (2.1) that

$$p(n \mid \text{odd parts}) = p(n \mid \text{distinct parts}) \text{ for } n \geq 1.$$

To prove Euler's identity using generating functions, we consider the related generating functions:

$$\sum_{n=0}^{\infty} p(n \mid \text{odd parts}) q^n = \prod_{n \text{ odd}} \frac{1}{1-q^n}$$

and, by eq. (3.2),

$$\sum_{n=0}^{\infty} p(n \mid \text{distinct parts}) q^n = \prod_{n=1}^{\infty} (1+q^n).$$

Now,

$$\begin{aligned}
\prod_{n=1}^{\infty} (1+q^n) &= (1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)\cdots \\
&= \left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^4}{1-q^2}\right)\left(\frac{1-q^6}{1-q^3}\right)\left(\frac{1-q^8}{1-q^4}\right)\left(\frac{1-q^{10}}{1-q^5}\right)\cdots \\
&= \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots} \\
&= \prod_{n \text{ odd}} \frac{1}{1-q^n}.
\end{aligned}$$

Here we have proven a combinatorial result through the manipulation of equations.

3.2 Gaussian Polynomials

In this section we will introduce properties of polynomials in q called Gaussian polynomials (also called q-binomial numbers or q-binomial coefficients) that will be helpful in proving the Rogers-Ramanujan identities.

The q-binomial numbers are q-analogs of the binomial numbers. The well-known binomial numbers have the following formula:

$$\binom{N}{m} = \frac{N!}{m!(N-m!)} \quad \text{for } N \geq m \geq 0.$$

This can be rewritten as

$$\binom{N}{m} = \frac{N(N-1)(N-2)\cdots(N-m+1)}{m(m-1)(m-2)\cdots 1}.$$

The analogous formula for q-binomial numbers is

$$\begin{bmatrix} N \\ m \end{bmatrix} = \frac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N-m+1})}{(1-q^m)(1-q^{m-1})\cdots(1-q)}. \tag{3.4}$$

It can be shown using an argument with lattice paths and integer partitions that more generally, the q-binomial numbers can be defined equivalently as

$$\begin{bmatrix} N+m \\ m \end{bmatrix} = \sum_{n \geq 0} p(n \mid \leq m \text{ parts, each} \leq N) q^n.$$

This is a natural refinement such that at the limit $q \to 1$, we recover $\binom{N+m}{m}$. Many of the properties of the binomial numbers also carry over to the q-binomial numbers. For example, the q-binomial numbers have the symmetric property

$$\begin{bmatrix} N+m \\ m \end{bmatrix} = \begin{bmatrix} N+m \\ N \end{bmatrix}.$$

There exists a well-known recurrence relation for binomial numbers that can be written as

$$\binom{N+m}{m} = \binom{N+m-1}{m} + \binom{N+m-1}{m-1}$$

This recurrence can be extended to q-binomial numbers using the following argument with integer partitions:

$$\begin{aligned} p(n \mid \leq m \text{ parts, each} \leq N) q^n &= p(n \mid \leq m-1 \text{ parts, each} \leq N) q^n \\ &\quad + q^m p(n-m \mid \leq m \text{ parts, each} \leq N-1) q^{n-m}. \end{aligned}$$

Summation over n gives us the q-analog recurrence for q-binomial numbers:

$$\begin{bmatrix} N+m \\ m \end{bmatrix} = \begin{bmatrix} N+m-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} N+m-1 \\ N-1 \end{bmatrix}. \tag{3.5}$$

A similar argument gives us the alternate recurrence:

$$\begin{bmatrix} N+m \\ m \end{bmatrix} = q^N \begin{bmatrix} N+m-1 \\ m-1 \end{bmatrix} + \begin{bmatrix} N+m-1 \\ N-1 \end{bmatrix}. \tag{3.6}$$

Additionally, to prove the Rogers-Ramanujan Identities, we will need to know the limit formulas for q-binomial numbers. For fixed m, by eq. (3.4),

$$\begin{aligned} \lim_{N\to\infty} \begin{bmatrix} N \\ m \end{bmatrix} &= \lim_{N\to\infty} \frac{\prod_{j=1}^{N}(1-q^j)}{\prod_{j=1}^{m}(1-q^j)\prod_{j=1}^{N-m}(1-q^j)} \\ &= \frac{\prod_{j=1}^{\infty}(1-q^j)}{\prod_{j=1}^{m}(1-q^j)\prod_{j=1}^{\infty}(1-q^j)} \\ &= \frac{1}{\prod_{j=1}^{m}(1-q^j)}. \end{aligned} \tag{3.7}$$

Then, for fixed m_1 and m_2, with $R > S$ positive, by eq. (3.4),

$$\begin{aligned} \lim_{N\to\infty} \begin{bmatrix} RN+m_1 \\ SN+m_2 \end{bmatrix} &= \lim_{N\to\infty} \frac{\prod_{j=1}^{RN+m_1}(1-q^j)}{\prod_{j=1}^{SN+m_2}(1-q^j)\prod_{j=1}^{(R-S)N+m_1-m_2}(1-q^j)} \\ &= \frac{\prod_{j=1}^{\infty}(1-q^j)}{\prod_{j=1}^{\infty}(1-q^j)\prod_{j=1}^{\infty}(1-q^j)} \\ &= \frac{1}{\prod_{j=1}^{\infty}(1-q^j)}. \end{aligned} \tag{3.8}$$

4 The Rogers-Ramanujan Identities

Just like there is no simple bijection proving the Rogers-Ramanujan identities, there is no simple generating function proof of the Rogers-Ramanujan identities. However, using Gaussian polynomials and the following Jacobi's triple product identity, we can prove a polynomial version of the identities. We will only detail a proof overview of the Rogers-Ramanujan identities, and the full proof referenced can be found in [1]. Many other versions of this proof are available.

4.1 Jacobi's Triple Product Identity

The following identity can be proved by manipulating generating functions and is a powerful tool for proving many results about generating functions in q and z. We will use it in the proof overview for the Rogers-Ramanujan identities.

Theorem 1 (Jacobi's Triple Product Identity)**.**

$$\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2} = \prod_{n=1}^{\infty} (1-q^n)(1+zq^n)(1+z^{-1}q^{n-1}).$$

for $|q| < 1$, $z \neq 0$.

4.2 Generating Functions for Rogers-Ramanujan identities

As in the first Rogers-Ramanujan identity (1.1), we can immediately see that the generating function for parts congruent to $\pm 1 \pmod 5$ is

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})},$$

and as in the second identity (1.2), we have the generating function for parts congruent to $\pm 2 \pmod 5$ is

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})}.$$

For the left-hand-side of the identity, an arithmetic argument shows that for any partition of n with exactly m 2-distinct parts there exists a corresponding partition of $n-(1+3+\cdots+(2m-1)) = n-m^2$ into at most m parts. Summing over all values of m, we get that the generating function for all partitions with 2-distinct parts is

$$1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

Similarly, we can find a generating function in the case with 2-distinct parts which are all greater than 1,

$$1 + \sum_{m=1}^{\infty} \frac{q^{m^2+m}}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

Hence we have that the two Rogers-Ramanujan identities are equivalent to

$$1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1-q)(1-q^2)\cdots(1-q^m)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})}, \tag{4.1}$$

and

$$1+\sum_{m=1}^{\infty}\frac{q^{m^2+m}}{(1-q)(1-q^2)\cdots(1-q^m)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-3})(1-q^{5n-2})}. \tag{4.2}$$

4.3 Proof Overview of the Rogers-Ramanujan Identities

To prove the Rogers-Ramanujan identities, we will first define four sequences of polynomials:

$$s_n(q)=\sum_{j=0}^{\infty}q^{j^2}\begin{bmatrix}n\\j\end{bmatrix} \tag{4.3}$$

$$t_n(q)=\sum_{j=0}^{\infty}q^{j^2+j}\begin{bmatrix}n\\j\end{bmatrix} \tag{4.4}$$

$$\sigma_n(q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(5j+1)/2}\begin{bmatrix}2n\\n+2j\end{bmatrix} \tag{4.5}$$

$$\tau_n(q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(5j-3)/2}\begin{bmatrix}2n+1\\n+2j\end{bmatrix} \tag{4.6}$$

The proof can then be summarized by the following steps:

Step 1: Show $s_n(q)=s_{n-1}(q)+q^n t_{n-1}(q)$ using eq. (3.5)

Step 2: Show $t_n(q)-q^n s_n(q)=(1-q^n)t_{n-1}(q)$. using eq. (3.4).

Step 3: Use mathematical induction and the initial conditions $s_0(q)=t_0(q)=1$ to show that $s_n(q)$ and $t_n(q)$ are uniquely defined for all n.

Step 4: Show that $\sigma_n(q)$ and $\tau_n(q)$ follow the same recurrence relations as $s_n(q)$ and $t_n(q)$ in Step 1 and Step 2 using the recurrence equations (3.5) and (3.6).

Step 5: Observe that because $\sigma_n(q)$ and $\tau_n(q)$ have the same initial conditions as $s_n(q)$ and $t_n(q)$, $\sigma_0(q)=\tau_0(q)=1$, paired with identical recurrence relations, we must have that $s_n(q)=\sigma_n(q)$ and $t_n(q)=\tau_n(q)$.

Once these steps have been achieved, we can see that

$$\begin{aligned}1+\sum_{m=1}^{\infty}\frac{q^{m^2}}{(1-q)(1-q^2)\cdots(1-q^m)}&=\lim_{n\to\infty}s_n(q)=\lim_{n\to\infty}\sigma_n(q)\\&=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(5j+1)/2}\frac{1}{\prod_{m=1}^{\infty}(1-q^m)}\\&=\frac{\prod_{m=1}^{\infty}(1-q^{5m})(1-q^{5m-2})(1-q^{5m-3})}{\prod_{m=1}^{\infty}(1-q^m)}\\&=\prod_{m=1}^{\infty}\frac{1}{(1-q^{5m-4})(1-q^{5m-1})}\end{aligned}$$

using eq. (3.8) and Jacobi's triple product identity, Theorem 1, with q replaced by q^5 and z replaced by $-q^{-2}$. This results in the first Rogers-Ramanujan identity, eq. (4.1).

Similarly, we can see that

$$
\begin{aligned}
1+\sum_{m=1}^{\infty} \frac{q^{m^2+m}}{(1-q)(1-q^2)\cdots(1-q^m)} &= \lim_{n\to\infty} t_n(q) = \lim_{n\to\infty} \tau_n(q) \\
&= \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j-3)/2} \frac{1}{\prod_{m=1}^{\infty}(1-q^m)} \\
&= \frac{\prod_{m=1}^{\infty}(1-q^{5m})(1-q^{5m-1})(1-q^{5m-4})}{\prod_{m=1}^{\infty}(1-q^m)} \\
&= \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-4})(1-q^{5m-1})}
\end{aligned}
$$

using (3.8) and Jacobi's triple product identity with q replaced by q^5 and z replaced by $-q^{-1}$. This results in the second Rogers-Ramanujan identity, eq. (4.2).

References

[1] George Andrews and Kimmo Eriksson. *Integer Partitions*. Cambridge University, Cambridge, United Kingdom, 2004.